RADON TRANSFORM ON GRAPHS AND ADMISSIBLE COMPLEXES [1]

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Integral geometry on finite sets arises as a natural analogy of classical integral geometry – in the sense of Gelfand, see, for example, [3], [4], [5]. On one hand, it is a part of combinatorial analysis and, on the other hand, its ideas, problems, constructions etc. are taken from classical integral geometry. Possibly, first works on finite integral geometry were papers by Bolker, see, for example, [1], [2]. They considered the Radon transform in vector spaces over finite fields.

In this paper we study the Radon transform R on *graphs*. It assigns to a function f defined on vertices of a graph G its "integrals" over edges, i.e. a function Rf defined on edges whose value at an edge x is equal to the sum of values of f at ends of this edge.

We describe the kernel and the image of the transform R and write the inversion formula in the case when this transform is injective.

Further, we consider *complexes* in the graph G. Let the graph G have n vertices. A complex is a subgraph of G having n vertices and n edges. We give a characterization of *admissible* complexes, i.e. such that the restriction of the transform R to them is injective.

In classical integral geometry the notions of a complex and an admissible complex were introduced by Gelfand and Graev [3], [4], [5]. The study of complexes in $\mathbb{C}^n$, $\mathbb{R}^n$ and their applications was the subject of investigations for one of the authors of this paper (K), see, for example, [7], [8].

1 Preliminaries

In this Section we recall some facts from graph theory, here we rely on [6], and give some constructions.

A graph G consists from a finite nonempty set V of *vertices* and a set X of 2-subsets of V called *edges*. We write $G = (V, X)$. Thus, an edge $x \in X$ is an unordered pair of different vertices $u, v \in V$. One says that the edge x joins u and v and in this case we write $x = uv$. Therefore, we deal with the so-called *simple* graphs: no loops and no multiple edges.

A subgraph of the graph $G = (V, X)$ is a graph $G' = (V', X')$ such that $V' \subset V$, $X' \subset X$.

Let $G' = (V', X')$, $G'' = (V'', X'')$ be two subgraphs of the graph G. The union $G' \cup G''$ is the subgraph $(V' \cup V'', X' \cup X'')$, the intersection $G' \cap G''$ is the subgraph $(V' \cap V'', X' \cap X'')$.

A *sequence* A in a graph G is a sequence of vertices and edges:

$$u = v_0, x_1, v_1, x_2, \ldots, v_{m-1}, x_m, v_m = v \qquad (1.1)$$

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such that $x_i = v_{i-1}v_i, i = 1, \ldots, m$. We say that the sequence A connects the vertices u and v. The number m is called the length of the sequence, we denote it $\ell(A)$. The vertices v_0 and v_m are called the initial point (the beginning) and the end of the sequence A respectively.

Let us have two sequences: a sequence A going from u to v and a sequence B going from v to w. We call by the sum of the sequences A and B the sequence $A + B$ going from u to w by vertices and edges of the sequences A and B.

For the sequence A going from u to v, see (1.1), we call the opposite sequence and denote by $(-A)$ the sequence running vertices and edges (1.1) in reverse order from v to u, i.e. the sequence

$$v = v_m, x_m, v_{m-1}, \ldots, v_1, x_1, v_0 = u.$$

If the beginning and the end of a sequence coincide, then the sequence is called the *closed sequence*. If all vertices of a a sequence are distinct (except perhaps the first and the last), then this sequence is called a *path*. If the beginning and the end of a path coincide, then this path is called a *closed path*.

Let C be the closed path (1.1). We define a *simple circuit* (the circuit for brevity) to be the subgraph (not sequence !) Z of the graph G, consisting of vertices v_i and edges x_i occuring in (1.1). The length of a circuit is the number of edges in it.

A graph is called connected if any its vertices may be connected by a sequence. An arbitrary graph is the disjunct union of its connected components, maximal connected subgraphs of this graph.

The distance $d(u, v)$ between the vertices u, v is the length of the shortest path connecting u with v. The distance is a metric.

For the vertices u and v, consider some shortest path connecting u and v and denote by $[u, v]$ the *subgraph* consisting of vertices and edges of this path. We call this subgraph the *segment* connecting u with v. The segment connecting u with v is defined not uniquely.

Let G be a graph (V, X), let V and X have n and r elements respectively. The number

$$\chi(G) = n - r$$

is called the *Euler characteristic* of the graph G.

Let a graph G have s connected components. Then $\chi(G) \leqslant s$, so that for a connected graph G we have

$$\chi(G) \leqslant 1.$$

A connected graph without circuits is called a *tree*. A connected graph G is a tree if and only if

$$\chi(G) = 1. \tag{1.2}$$

Let u, v, w be three distinct vertices of the graph G. Let us call the subgraph $T = [u, v] \cup [v, u] \cup [u, w]$ the *triangle* with the vertices u, v, w. The triangle is defined not uniquely. But the *perimeter* of this triangle, i.e. the number

$$p(T) = p(u, v, w) = d(u, v) + d(v, w) + d(u, w), \tag{1.3}$$

is well-defined.

Let M be a finite set. By $|M|$ we denote the number of elements of this set. Denote by $L(M)$ the space of functions on M with the values in $\mathbb{C}$. The inner product of functions $f, g \in L(M)$ is the number

$$(f, g) = \sum_{x \in M} f(x)\overline{g(x)}.$$

The delta function $\delta_a(x)$ concentrated at a point $a \in M$ is the following function from $L(M)$:

$$\delta_a(x) = \begin{cases} 1, & x = a, \\ 0, & x \neq a. \end{cases}$$

All delta functions δ_a, $a \in M$, form a basis in $L(M)$.

For the sequence A in the graph $G = (V, X)$ given by (1.1), we introduce the following function from $L(X)$ ("alternating delta function"):

$$\varepsilon_A(x) = \sum_{i=1}^{m} (-1)^{i-1} \delta_{x_i}(x).$$

2 Radon transform on graphs

Let $G = (V, X)$ be a graph with n vertices and r edges.

We call the *Radon transform* on the graph G the operator $R : L(V) \to L(X)$ that to each function $f \in L(V)$ assigns the function $Rf \in L(X)$ whose value at an edge $x = uv$ is equal to the sum of values of the function f at the vertices u, v of this edge (an "integral" of the function f over the edge x):

$$(Rf)(x) = f(u) + f(v), \quad x = uv. \tag{2.1}$$

The Radon transform on a graph G commutes with the restriction to its subgraphs. Namely, let $G' = (V', X')$ be a subgraph of the graph G, let R' be the Radon transform on G'. Denote by f' and φ' the restrictions of functions $f \in L(V)$ and $\varphi \in L(V)$ to V' and X' respectively. Then

$$R'f' = (Rf)'.$$

Therefore, we may denote for brevity the Radon transforms on a graph G and on its subgraphs by the same letter R.

We want to describe the kernel $\operatorname{Ker} R$ and the image $\operatorname{Im} R$ of the operator R. In particular, we want to know, when $\operatorname{Ker} R = \{0\}$, i.e. R is injective, and, if so, to write the inversion formula.

For that it is sufficient to consider that the graph G is *connected.*

One of tools is the inner product of functions Rf and ε_A. Let a sequence A of the length $\ell(A)$ connect the vertex u with the vertex v, see (1.1). The inner product of the function Rf and the function ε_A is equal to

$$(Rf, \varepsilon_A) = \sum_{i=1}^{m} (-1)^{i-1} (Rf)(x_i). \tag{2.2}$$

Lemma 2.1 *We have*

$$(Rf, \varepsilon_A) = f(u) - (-1)^{\ell(A)} f(v). \tag{2.3}$$

In particular, let A be a closed sequence. If its length is even, then

$$(Rf, \varepsilon_A) = 0, \tag{2.4}$$

and if its length is odd, then

$$(Rf, \varepsilon_A) = 2f(u). \tag{2.5}$$

Proof. In virtue of (2.1) the values of the function f at vertices between u and v in the sum (2.2) are annihilated. □

Lemma 2.2 *If the graph G is connected, then the dimension of the kernel* $\operatorname{Ker} R$ *is not greater than one:*

$$\dim \operatorname{Ker} R \leqslant 1. \tag{2.6}$$

Proof. Let $f \in \operatorname{Ker} R$. Fix a vertex $v \in G$. Then at adjacent vertices u, i.e. such that $d(u,v) = 1$, the function f must have the value $(-f(v))$, so that $f(u) = -f(v)$ for $d(u,v) = 1$. For an arbitrary $u \in G$, we obtain by iteration:

$$f(u) = (-1)^{d(u,v)} f(v). \tag{2.7}$$

Therefore, values of the function f at the vertices $u \in G$ are completely defined by its value at a fixed vertex. □

We say that a connected graph G has *class* $c = 0$ or $c = 1$, if $\dim \operatorname{Ker} R = c$.
If $c = 0$, then the operator R is injective.

Theorem 2.3 *A connected graph G has class* 0 *(i.e. R is injective), if and only if there exists a triangle in the graph G with odd perimeter. A connected graph G has class* 1*, if and only if each triangle in the graph G has even perimeter.*

Proof. Let $f \in \operatorname{Ker} R$. Fix a vertex v. The values of f at vertices u are given by formula (2.7). These values must not depend upon the choice of the initial point v. Let us take some other initial point w. Then, according to (2.7), we have

$$\begin{aligned} f(u) &= (-1)^{d(u,w} f(w) \\ &= (-1)^{d(u,w}(-1)^{d(v,w} f(v). \end{aligned} \tag{2.8}$$

Comparing (2.7) and (2.8), we obtain, that f can be not zero if and only if

$$d(u,v) \equiv d(u,w) + d(v,w),$$

here and further the sign $\equiv$ denotes the congruence modulo 2. Therefore,

$$d(u,v) + d(u,w) + d(v,w) \equiv 0,$$

which means that the perimeter of a triangle with the vertices u, v, w is even. □

In particular, in a *tree* each triangle has an even perimeter, so that a tree has class 1, the operator R on it is not injective.

Theorem 2.4 *A connected graph G has class* 0 *(i.e. the operator R is injective), if and only if there exists a circuit in the graph G of odd length. A connected graph G has class* 1 *if and only if each circuit in the graph G has even length.*

Proof. Let a connected graph G have a circuit Z of odd length with consecutive vertices $v_1, v_2, \ldots, v_{2k+1}$. Consider $2k-1$ triangles $T_1, T_2, \ldots, T_{2k-1}$: a triangle T_i has the vertices v_1, v_{i+1}, v_{i+2}. The perimeter p_i of the triangle T_i is

$$p_i = d(v_1, v_{i+1}) + 1 + d(v_1, v_{i+2}). \tag{2.9}$$

Let us summarize (2.9) over $i = 1, \ldots, 2k-1$. We obtain

$$\begin{aligned} p_1 + \ldots + p_{2k-1} &= d(v_1, v_2) + 2\big[d(v_1, v_3) + \ldots + d(v_1, v_{2k})\big] + d(v_1, v_{2k+1}) + 2k - 1 \\ &= 2k + 1 + 2\big[d(v_1, v_3) + \ldots + d(v_1, v_{2k})\big]. \end{aligned}$$

From this we have

$$p_1 + \ldots + p_{2k-1} \equiv 1.$$

Hence, at least one of the triangles T_i has odd perimeter. By Theorem 2.3 we obtain $c = 0$.

Inversely, let a connected graph G contain a triangle T with vertices u, v, w having odd perimeter p, i.e.

$$p \equiv 1, \tag{2.10}$$

see (1.3). Let us construct a circuit Z with odd length.

Condition (2.10) implies that the intersection of three segments $[u,v]$, $[v,w]$, $[u,w]$ is empty. Indeed, if this intersection contains a vertex a, then

$$\begin{aligned} d(u,v) &= d(u,a) + d(a,v), \\ d(v,w) &= d(v,a) + d(a,w), \\ d(u,w) &= d(u,a) + d(a,w). \end{aligned}$$

Summarizing, we obtain that the perimeter p is equal to

$$2\big[d(u,a) + d(v,a) + d(w,a)\big],$$

hence is even. It contradicts to (2.10).

Let u' be a vertex in the intersection $[u,v] \cap [u,w]$, the most distant from u (it may coincide with u). Similarly we define vertices $v' = [u,v] \cap [v,w]$ and $w' = [u,w] \cap [v,w]$. All vertices u', v', w' are different, since the intersection $[u,v] \cap [v,w] \cap [u,w]$ is empty. Let segments $[u',v']$, $[v',w']$, $[u',w']$ be parts of the segments $[u,v]$, $[v,w]$, $[u,w]$, respectively. The union Z of these segments (it is the triangle) is a circuit (since $[u',v']$ and $[v',w']$ intersect only at one vertex v' etc.).

The lengths of these segments are as follows:

$$\begin{aligned} d(u',v') &= d(u,v) - d(u,u') - d(v,v'), \\ d(v',w') &= d(v,w) - d(v,v') - d(w,w'), \\ d(u',w') &= d(u,w) - d(u,u') - d(w,w'). \end{aligned}$$

Summarizing, we obtain, that the perimeter p' of the triangle Z is

$$p' = p - 2\big[d(u,u') + d(v,v') + d(w,w')\big].$$

Hence, $p' \equiv p$, so that $p' \equiv 1$, which means that the length of the circuit Z is odd. □

Let us write the inversion formula for the Radon transform in the case $c = 0$. In this case a connected graph G contains a circuit Z of an odd length (Theorem 2.4). Let us take an arbitrary vertex u in the graph G. Let us consider the following closed sequence A, that begins and ends at the vertex u. Let P be a sequence going from the vertex u to some vertex $v \in Z$ (such sequence can be absent, if $u \in Z$). Let C be a closed path going along the circuit Z from the vertex v to v itself. Its length $\ell(C)$ is equal to the length $\ell(Z)$, hence is odd. We set $A = P + C - P$. The length of the sequence A is equal to

$$\ell(A) = \ell(P) + \ell(C) + \ell(P) = \ell(C) + 2\ell(P),$$

so is odd. By formula (2.5) we obtain

$$f(u) = \frac{1}{2}(Rf, \varepsilon_A).$$

Remark. Formula (2.3) is a kind of "the Newton-Leibniz formula", therefore, the Radon transform R is a kind of "differentiation" (inspite of the fact that it was defined as an "integral"). Let $\varphi \in \operatorname{Im} R$, let f be its "primitive", i.e. such a function that $Rf = \varphi$. Fix a vertex $v \in G$. Then the value of the function f at an arbitrary vertex u is given by the formula (see (2.3)):

$$f(u) = (Rf, \varepsilon_A) + (-1)^{\ell(A)} f(v),$$

where A is a sequence going from the vertex v to the vertex u.

For $c = 1$ the value $f(v)$ may be arbitrary, so that f is defined up to a function in $\operatorname{Ker} R$ ("constants"), and for $c = 0$ the value of $f(v)$ is defined uniquely.

3 The image of the Radon transform

Let $G = (V, X)$ be a connected graph with n vertices and r edges. It has class $c = 0, 1$, see Section 2. The image $\operatorname{Im} R \subset L(X)$ of the Radon transform has dimension $n - c$. We want to describe the subspace $\operatorname{Im} R$ in $L(X)$. For that, it is sufficient to give the description of the kernel $\operatorname{Ker} R^*$ of the conjugate operator R^*.

This operator R^* is defined by the condition

$$(Rf, \varphi) = (f, R^*\varphi), \tag{3.1}$$

where $f \in L(V)$, $\varphi \in L(X)$ and the inner products are taken in $L(V)$ and $L(X)$. It acts by the formula:

$$(R^*\varphi)(u) = \sum_{u \in x} \varphi(x).$$

It follows from (3.1) that $\operatorname{Im} R$ is the orthogonal complement to the kernel $\operatorname{Ker} R^*$ in $L(X)$. This kernel $\operatorname{Ker} R^*$ has dimension

$$\dim \operatorname{Ker} R^* = -\chi(G) + c, \tag{3.2}$$

where, recall, $\chi(G)$ is the Euler characteristic ($\chi(G) = n - r$) of the graph G. Indeed, this dimension is equal to $r - (n - c) = -(n - r) + c$.

Let us show some functions in $\operatorname{Ker} R^*$.

Each circuit Z of even length gives a function φ in $\operatorname{Ker} R^*$, namely, let C be a closed path passing along Z, then $\varphi = \varepsilon_C$. Indeed, by (2.4) ε_C is orthogonal to $\operatorname{Im} R$.

Each pair of different circuits Z and W of odd length generates a function ψ in $\operatorname{Ker} R^*$ in the following way. Let C be a closed path passing along Z from a vertex $u \in Z$ to itself and D a closed path passing along W from a vertex $v \in W$ to itself. Lengths $\ell(C)$ and $\ell(D)$ are odd. Let P be a sequence going from the vertex u to the vertex v. Let us consider the closed sequence $A = C+P+D-P$ going from u to u. Its length $\ell(A)$ is equal to $\ell(C)+\ell(D)+2\ell(P)$, so is even. We set $\psi = \varepsilon_A$.

Now we present a basis in $\operatorname{Ker} R^*$.

First we construct a family of circuits in the graph G.

Let Z_1 be some circuit in the graph G. Remove some edge of Z_1 from G. We obtain the connected graph G_1, it has n vertices and $r-1$ edges, so that its Euler characteristic is $\chi(G)+1$. Let us take in G_1 some circuit Z_2 and remove an edge of Z_2 from G_1. We obtain the connected graph G_2 with Euler characteristic $\chi(G)+2$ and so on. After k steps we obtain the connected graph G_k with Euler characteristic $\chi(G)+k$, which contains no circuits, i.e. G_k is a tree. By (1.5) we obtain $\chi(G)+k=1$, whence

$$k = -\chi(G) + 1. \tag{3.3}$$

Let us call these circuits $Z_1, \ldots, Z_k$ *basis* circuits. This family of circuits contains p circuits of even length and q circuits of odd length, $p+q=k$, p, $q \geqslant 0$.

Let $q=0$. Then all the circuits $Z_1, \ldots, Z_k$ have even length, each of them produces a function in $\operatorname{Ker} R^*$, see above. We obtain functions $\varphi_1, \ldots, \varphi_k$ in $\operatorname{Ker} R^*$. They are linearly independent, therefore,

$$k \leqslant \dim \operatorname{Ker} R^*. \tag{3.4}$$

On the other hand, comparing (3.2) and (3.3) we see:

$$k - \dim \operatorname{Ker} R^* = 1 - c,$$

and by (3.4) and (2.6) we have

$$0 \geqslant k - \dim \operatorname{Ker} R^* = 1 - c \geqslant 0,$$

whence $c=1$ and $k = \dim \operatorname{Ker} R^*$. Therefore, the functions $\varphi_1, \ldots, \varphi_k$ form a basis in $\operatorname{Ker} R^*$.

Let $q \geqslant 1$. Then $c=0$. Let $Z_1, \ldots, Z_q$ be all basis circuits of odd length. Consider $q-1$ pairs of circuits: $(Z_1, Z_2), \ldots, (Z_1, Z_q)$. These pairs produce $q-1$ functions $\psi_1, \ldots, \psi_{q-1}$ in $\operatorname{Ker} R^*$, as it was pointed above. The remained p circuits of even length give p functions $\varphi_1, \ldots, \varphi_p$ in $\operatorname{Ker} R^*$, see above. Altogether we obtain $q-1+p=k-1$, i.e. $-\chi(G)$, functions in $\operatorname{Ker} R^*$. They are linearly independent. In virtue of (3.2) they form a basis in $\operatorname{Ker} R^*$.

The bases in $\operatorname{Ker} R^*$, just constructed, give relations for functions in $\operatorname{Im} R$. These relations have the form

$$(Rf, \varepsilon_A) = 0,$$

where A is a closed sequence constructed as it was said above either by a circuit of even length or by a pair of circuits of odd length.

4 Admissible complexes

Let $G = (V, X)$ be a graph with n vertices and r edges, let $n \leqslant r$, so $\chi(G) \leqslant 0$.

By analogy with [4], [5] we define a *complex* in the graph G to be a subgraph K of G which has n vertices and n edges. Therefore, $K = (V, Y)$, where $Y \subset X$ and $\chi(K) = 0$.

By analogy with [4], [5] again, we call a complex K *admissible*, if the Radon transform $R : L(V) \to L(Y)$ on K is injective, therefore, the Radon transform R on K is an isomorphism.

Notice that any connected component of an admissible complex can not be a tree.

Theorem 4.1 *A complex K is admissible if and only if each connected component of K has a unique circuit and this circuit has odd length.*

Proof. Let each connected K_i component of K have a circuit of odd length. Then by Theorem 2.4 the Radon transform R on K_i is injective, therefore, the Radon transform R on K is also injective, so that K is admissible.

Now let a complex K of the graph G be admissible. Let $K_1, \ldots, K_s$ be its connected components. Denote by n_i the number of vertices of K_i. For each K_i the Radon transform R on K_i is injective, hence K_i is not a tree, so that $\chi(K_i) \leqslant 0$. Since $\chi(K) = 0$, we have

$$\chi(K_1) + \ldots + \chi(K_s) = 0.$$

Hence, $\chi(K_i) = 0$ for all $i = 1, \ldots, s$. Therefore, the number of edges in the graph K_i is equal to n_i too. Let Z_i be some circuit in K_i. Remove from K_i one edge of Z_i. We obtain a connected graph K_i' with n_i vertices and $n_i - 1$ edges, so $\chi(K_i') = 1$, so that K_i' is a tree. Therefore, Z_i is a single circuit in K_i. By Theorem 2.4 this circuit Z_i has odd length. □

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